CALCULUS

Differentiation

Introduction :- The word calculus stands for method of computation. The most common use of calculus is in regard to the computation of the rate of change in one variable with reference to an very small change in the other variable. calculus gives the technique for measuring these changes in the dependent variable with reference to a very small change, approaching almost zero, in the independent variable (or variables). The techniques concerning the calculation of the average rate of change are studied under differentiation.

Ex :- The demand function it would be possible to find the degree of change in demand with reference to a small change in price (OR) Income (OR) both as the case may be and also the maximum and minimum values of the function.

Definition :- (Differentiation (OR) Derivative)

A derivative is the limit of the ratio of the increment in the function corresponding to a small increment in the argument as tends to zero.

Let $y = f(x)$ be a function (Here 'x' is independent variable) $\rightarrow$ ①

Let δx be an increment in the value of 'x' and δy be an increment in the value of 'y'

So from ① it becomes

$$y + \delta y = f(x + \delta x)$$

$$\delta y = f(x+\delta x) - y$$

$$\delta y = f(x+\delta x) - f(x) \quad (\because y = f(x))$$

Divide on b/s by δx

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Take limit $\delta x \to 0$ on both sides

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

$\left(\because \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}\right)$

$$\therefore \boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} \rightarrow ② \quad (\because \text{Take } \delta x = h)$$

NOTE :- ① If $y = f(x)$ then $\frac{dy}{dx}$ means ~~derivative (Differe~~ derivative (Differential coefficient) of 'y' with respect to 'x'.

② $\frac{dy}{dx}$ also written as y' (OR) $f'(x)$.

③ From above equation ② is also called the first principle of derivative.

Formulae :

1. $\frac{d}{dx}(x^n) = n.x^{n-1}$, where $x > 0$

2. $\frac{d}{dx}(ax+b)^n = n.(ax+b)^{n-1}.a$

3. $\frac{d}{dx}(x) = 1$

4. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

5. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

6. $\frac{d}{dx}(K) = 0$ (Here K = Constant)

7. $\frac{d}{dx}[K.f(x)] = K.\frac{d}{dx}f(x).$

8. $\frac{d}{dx}[u(x) \pm v(x)] = \frac{d}{dx}u(x) \pm \frac{d}{dx}v(x)$

9. $\frac{d}{dx}(e^x) = e^x$

10. $\frac{d}{dx}(e^{Kx}) = K.e^{Kx}$

11. $\frac{d}{dx}(a^x) = a^x.\log a$

12. $\frac{d}{dx}(\log x) = \frac{1}{x}$

13. **Derivative of Product of Two functions** :- u, v are any functions in 'x'.

$$\boxed{\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v.\frac{du}{dx}}$$

(OR)

$$\frac{d}{dx}(u.v) = u.v' + v.u'$$

(Here $v' = \frac{dv}{dx}$, $u' = \frac{du}{dx}$)

(14) Derivative of Quotient of two functions :- u, v are functions of 'x' and v ≠ 0 then

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}}$$

(OR)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Here 'u' is the first function and 'v' is second function.

Problems

(1) If $y = x^7$ then find $\frac{dy}{dx}$

(OR)

Find the differential coefficient of the function x^7

Sol:- $y = x^7$.

Differentiate with respect to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^7)$$

$$= 7 \cdot x^{7-1} \qquad \left[\because \frac{d}{dx}(x^n) = n x^{n-1}\right]$$

$$\boxed{\frac{dy}{dx} = 7 \cdot x^6}$$

(2) Find the differential coefficient of the function x^{-4}

Sol:- Let $y = x^{-4}$

D. w. r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-4})$$

$$= -4 \cdot x^{-4-1} \qquad \left[\because \frac{d}{dx} x^n = n x^{n-1}\right]$$

$$\boxed{\frac{dy}{dx} = -4 x^{-5}}$$

3. If $y = 8x^3$ then find $\frac{dy}{dx}$

(OR)

Find the differential coefficient of $8x^3$

Sol:- $y = 8x^3$

D.w.r.to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(8x^3)$$

$$= 8\frac{d}{dx}(x^3) \quad \left[\because \frac{d}{dx}(K.f(x)) = K.\frac{d}{dx}f(x)\right]$$

$$= 8(3.x^{3-1})$$

$$= 8(3x^2)$$

$$\boxed{\frac{dy}{dx} = 24x^2}$$

4. If $y = 2020$ then find $\frac{dy}{dx}$.

Sol:- $y = 2020$

D.w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(2020)$$

$$\boxed{\frac{dy}{dx} = 0} \quad \left[\because \frac{d}{dx}(\underset{\text{Constant}}{K}) = 0\right]$$

5. Find the differential coefficient of $y = 4x^2 + 6x - 9$

Sol:- $y = 4x^2 + 6x - 9$

D.w.r.to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2 + 6x - 9)$$

$$= \frac{d}{dx}(4x^2) + \frac{d}{dx}(6x) - \frac{d}{dx}(9)$$

$$= 4\frac{d}{dx}(x^2) + 6\frac{d}{dx}(x) - \frac{d}{dx}(9)$$

$$= 4(2x) + 6(1) - 0$$

$$\boxed{\frac{dy}{dx} = 8x + 6}$$

⑥ If $y = x^{5/3}$ then find $\frac{dy}{dx}$.

Sol:-

$$y = x^{5/3}$$

D.w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{5/3}\right)$$

$$= \frac{5}{3}\, x^{\frac{5}{3}-1} \qquad \left(\because \frac{d}{dx}(x^n) = n.x^{n-1}\right)$$

$$= \frac{5}{3}\, x^{\frac{5-3}{3}} = \frac{5}{3}\, x^{2/3}$$

$$\boxed{\frac{dy}{dx} = \frac{5}{3}\, x^{2/3}}$$

⑦ If $y = \frac{3}{x^5}$ then find $\frac{dy}{dx}$

Sol:-

$$y = \frac{3}{x^5}$$

$$y = 3.\frac{1}{x^5}$$

$$y = 3.x^{-5}$$

D.w.r.to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(3.x^{-5}\right)$$

$$= 3\frac{d}{dx}\left(x^{-5}\right)$$

$$= 3\left[-5\, x^{-5-1}\right]$$

$$= 3\left[-5\, x^{-6}\right]$$

$$\boxed{\frac{dy}{dx} = -15x^{-6}}$$

⑧ If $y = e^{2x} + 3\log x$ then find $\frac{dy}{dx}$

Sol:-

$$y = e^{2x} + 3\log x$$

D.w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{2x} + 3\log x\right)$$

$$= \frac{d}{dx} e^{2x} + \frac{d}{dx}(3\log x)$$

$$\frac{dy}{dx} = 2.e^{2x} + 3\frac{d}{dx}\log x = 2e^{2x} + 3\left(\frac{1}{x}\right) = 2e^{2x} + \frac{3}{x}.$$

7. Find the differential coefficient of $9x^4-7x^3+8x^2-\frac{8}{x}+\frac{10}{x^3}$.

Sol:- Let $y = 9x^4-7x^3+8x^2-\frac{8}{x}+\frac{10}{x^3}$

D. w. r. to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(9x^4-7x^3+8x^2-\frac{8}{x}+\frac{10}{x^3}\right)$$

$$= 9\frac{d}{dx}(x^4) - 7\frac{d}{dx}(x^3) + 8\frac{d}{dx}(x^2) - 8\frac{d}{dx}\left(\frac{1}{x}\right) + 10\frac{d}{dx}\left(\frac{1}{x^3}\right)$$

$$= 9\frac{d}{dx}(x^4) - 7\frac{d}{dx}(x^3) + 8\frac{d}{dx}(x^2) - 8\frac{d}{dx}(x^{-1}) + 10\frac{d}{dx}(x^{-3})$$

$$= 9(4x^3) - 7(3x^2) + 8(2x) - 8(-1.x^{-1-1}) + 10(-3x^{-3-1})$$

$$= 36x^3 - 21x^2 + 16x + 8x^{-2} - 30x^{-4}$$

$$\boxed{\frac{dy}{dx} = 36x^3 - 21x^2 + 16x + \frac{8}{x^2} - \frac{30}{x^4}}$$

8. Find the differential coefficient of $(5x+7)^{10}$

Sol:- Let $y = (5x+7)^{10}$

D. w. r. to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(5x+7)^{10}$$

$$= 10(5x+7)^{10-1} \cdot 5 \qquad \left[\because \frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1} \cdot a\right]$$

$$\boxed{\frac{dy}{dx} = 50(5x+7)^9}$$

9. Find the differential coefficient of the function

$$\frac{3x^7+x^5-2x^4+x-3}{x^4}$$

Sol:- $y = \frac{3x^7+x^5-2x^4+x-3}{x^4}$

$$= \frac{3x^7}{x^4} + \frac{x^5}{x^4} - \frac{2x^4}{x^4} + \frac{x}{x^4} - \frac{3}{x^4}$$

$$y = 3x^3 + x - 2 + \frac{1}{x^3} - \frac{3}{x^4}$$

$$y = 3x^3 + x - 2 + x^{-3} - 3x^{-4}$$

D.w.r to 'x'.

$$\frac{dy}{dx} = \frac{d}{dx}(3x^3 + x - 2 + x^{-3} - 3x^{-4})$$

$$= \frac{d}{dx}(3x^3) + \frac{d}{dx}(x) - \frac{d}{dx}(2) + \frac{d}{dx}(x^{-3}) - \frac{d}{dx}(3x^{-4})$$

$$= 3\frac{d}{dx}(x^3) + \frac{d}{dx}(x) - \frac{d}{dx}(2) + \frac{d}{dx}(x^{-3}) - 3\frac{d}{dx}(x^{-4})$$

$$= 3(3x^2) + 1 - 0 + (-3x^{-3-1}) - 3(-4x^{-4-1})$$

$$= 9x^2 + 1 - 3x^{-4} + 12x^{-5}$$

$$\frac{dy}{dx} = 9x^2 + 1 - \frac{3}{x^4} + \frac{12}{x^5}$$

(10) Find the differential coefficient of $\left(x - \frac{1}{x}\right)^2$.

Sol:- $y = \left(x - \frac{1}{x}\right)^2$

$$y = x^2 + \frac{1}{x^2} - 2(x)\left(\frac{1}{x}\right)$$

$$y = x^2 + \frac{1}{x^2} - 2$$

D.w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + \frac{1}{x^2} - 2\right)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}\left(\frac{1}{x^2}\right) - \frac{d}{dx}(2)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(x^{-2}) - \frac{d}{dx}(2)$$

$$= 2x + (-2x^{-2-1}) - 0$$

$$= 2x - 2x^{-3}$$

$$\boxed{\frac{dy}{dx} = 2x - \frac{2}{x^3}}$$

(11) find the differential coefficient of $(x+1)(3x-7)$

Sol:- $y = (x+1)(3x-7)$

let $u = x+1, \ V = 3x-7$

$$\boxed{\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x+1)(3x-7)\right]$$

$$= (x+1)\frac{d}{dx}(3x-7) + (3x-7)\frac{d}{dx}(x+1)$$

$$= (x+1)\left[\frac{d}{dx}3x - \frac{d}{dx}7\right] + (3x-7)\left[\frac{d}{dx}x + \frac{d}{dx}1\right]$$

$$= (x+1)\left[3\frac{d}{dx}(x) - \frac{d}{dx}7\right] + (3x-7)\left[\frac{d}{dx}x + \frac{d}{dx}1\right]$$

$$= (x+1)\left[3(1) - 0\right] + (3x-7)\left[1+0\right]$$

$$= (x+1)[3] + (3x-7)$$

$$= 3x+3+3x-7$$

$$\boxed{\frac{dy}{dx} = 6x-4}$$

(12) find the differential coefficient of $(x^2-4x+5)(x^3-2)$

Sol:- let $y = (x^2-4x+5)(x^3-2)$

D.w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x^2-4x+5)(x^3-2)\right]$$

let $u = x^2-4x+5, \ V = x^3-2.$ $\left[\because \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$

$$\frac{dy}{dx} = (x^2-4x+5)\frac{d}{dx}(x^3-2) + (x^3-2)\frac{d}{dx}(x^2-4x+5)$$

$$= (x^2-4x+5)\left(\frac{d}{dx}x^3 - \frac{d}{dx}2\right) + (x^3-2)\left(\frac{d}{dx}x^2 - \frac{d}{dx}4x + \frac{d}{dx}5\right)$$

$$= (x^2-4x+5)(3x^2-0) + (x^3-2)(2x-4+0)$$

$$= 3x^2(x^2-4x+5) + (x^3-2)(2x-4)$$

$$\frac{dy}{dx} = (3x^4 - 12x^3 + 15x^2) + (2x^4 - 4x^3 - 4x + 8)$$

$$\frac{dy}{dx} = 5x^4 - 16x^3 + 15x^2 - 4x + 8$$

(13) Find the differential coefficient of $(3x^2+1)(x^3+2x)$.

Sol:- Let $y = (3x^2+1)(x^3+2x)$

D. w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left[(3x^2+1)(x^3+2x)\right]$$

Let $u = 3x^2+1$, $v = x^3+2x$

$$\left[\because \frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\frac{dy}{dx} = (3x^2+1)\frac{d}{dx}(x^3+2x) + (x^3+2x)\frac{d}{dx}(3x^2+1)$$

$$= (3x^2+1)\left(\frac{d}{dx}x^3 + \frac{d}{dx}2x\right) + (x^3+2x)\left(\frac{d}{dx}3x^2 + \frac{d}{dx}1\right)$$

$$= \left[(3x^2+1)(3x^2+2)\right] + \left[(x^3+2x)(6x+0)\right]$$

$$= \left[9x^4 + 9x^2 + 2\right] + \left[6x^4 + 12x^2\right]$$

$$\boxed{\frac{dy}{dx} = 15x^4 + 21x^2 + 2}$$

(14) Find the differential coefficient of $\frac{x^2-4}{x-2}$

Sol:- Let $y = \frac{x^2-4}{x-2}$

$$y = \frac{x^2-2^2}{x-2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$y = x+2.$$

D. w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x+2)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(2)$$

$$= 1 + 0$$

$$\boxed{\frac{dy}{dx} = 1}$$

15. Find the differential coefficient of $\frac{2x+5}{3x-2}$

Sol:- Let $y = \frac{2x+5}{3x-2}$

D.w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x+5}{3x-2}\right)$$

let $u = 2x+5$, $v = 3x-2$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\frac{dy}{dx} = \frac{(3x-2)\frac{d}{dx}(2x+5) - (2x+5)\frac{d}{dx}(3x-2)}{(3x-2)^2}$$

$$= \frac{(3x-2)\left[\frac{d}{dx}2x + \frac{d}{dx}5\right] - (2x+5)\left[\frac{d}{dx}3x - \frac{d}{dx}2\right]}{(3x-2)^2}$$

$$= \frac{(3x-2)[2+0] - (2x+5)[3-0]}{(3x-2)^2}$$

$$= \frac{[2(3x-2)] - [3(2x+5)]}{(3x-2)^2} = \frac{[6x-4]-[6x+15]}{(3x-2)^2}$$

$$= \frac{6x-4-6x-15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-19}{(3x-2)^2}}$$

16. Find the differential coefficient of $\frac{x^2+2x+5}{x^2+2x+4}$

Sol:- $y = \frac{x^2+2x+5}{x^2+2x+4}$

D.w.r to 'x'

Let $u = x^2+2x+5$, $v = x^2+2x+4$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+2x+5}{x^2+2x+4}\right) \qquad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\frac{dy}{dx} = \frac{(x^2+2x+4)\frac{d}{dx}(x^2+2x+5) - (x^2+2x+5)\frac{d}{dx}(x^2+2x+4)}{(x^2+2x+4)^2}$$

$$= \frac{(x^2+2x+4)\left[\frac{d}{dx}x^2+\frac{d}{dx}2x+\frac{d}{dx}5\right] - (x^2+2x+5)\left[\frac{d}{dx}x^2+\frac{d}{dx}2x+\frac{d}{dx}4\right]}{(x^2+2x+4)^2}$$

$$= \frac{(x^2+2x+4)[2x+2] - (x^2+2x+5)[2x+2]}{(x^2+2x+4)^2}$$

$$= \frac{[2x^3+2x^2+4x^2+4x+8x+8] - [2x^3+2x^2+4x^2+4x+10x+10]}{(x^2+2x+4)^2}$$

$$\frac{dy}{dx} = \frac{[-2x-2]}{(x^2+2x+4)^2}$$

(17) Find the differential coefficient of $(5x^2+2x+3)^4$

Sol:-

$$y = (5x^2+2x+3)^4$$

D.w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(5x^2+2x+3)^4$$

$$= 4(5x^2+2x+3)^{4-1} \cdot \frac{d}{dx}[5x^2+2x+3]$$

$$= 4(5x^2+2x+3)^3\left[\frac{d}{dx}5x^2+\frac{d}{dx}2x+\frac{d}{dx}3\right]$$

$$\left[\because \frac{d}{dx}x^n = n.x^{n-1}\cdot\frac{d}{dx}(x)\right]$$

(OR)

$$\frac{d}{dx}\text{象}^n = n\,\text{象}^{n-1}\frac{d}{dx}\text{象}$$

$$= 4(5x^2+2x+3)^3\left[5\frac{d}{dx}x^2+2\frac{d}{dx}x+\frac{d}{dx}3\right]$$

$$= 4(5x^2+2x+3)^3[5(2x)+2(1)+0]$$

$$= 4(5x^2+2x+3)^3[10x+2]$$

$$= 4(10x+2)(5x^2+2x+3)^3$$

$$\frac{dy}{dx} = (40x+8)(5x^2+2x+3)^3$$

(18) Find the differential coefficient of $\frac{1}{3x^2-2x+7}$.

Sol:- $y = \frac{1}{3x^2-2x+7}$

$y = (3x^2-2x+7)^{-1}$

D.w.r to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2-2x+7)^{-1}$$

$$= (-1)(3x^2-2x+7)^{-1-1}\frac{d}{dx}(3x^2-2x+7)$$

$$= (-1)(3x^2-2x+7)^{-2}\left[\frac{d}{dx}3x^2 - \frac{d}{dx}2x + \frac{d}{dx}7\right]$$

$$= (-1)(3x^2-2x+7)^{-2}\left[3\frac{d}{dx}x^2 - 2\frac{d}{dx}x + \frac{d}{dx}7\right]$$

$$= (-1)\frac{1}{(3x^2-2x+7)^2}\left[3(2x) - 2(1) + 0\right]$$

$$= \frac{-1}{(3x^2-2x+7)^2}\left[6x-2\right]$$

$$\frac{dy}{dx} = \frac{-(6x-2)}{(3x^2-2x+7)^2}$$

→ **Derivative of Implicit function**: Some times 'y' is not given directly in terms of 'x' the value of $\frac{dy}{dx}$ can be calculated by differentiating term by term and then seperating $\frac{dy}{dx}$.

(19) Find $\frac{dy}{dx}$ if $x^3+y^3 = 3axy$

Sol:- $x^3+y^3 = 3axy$

D.w.r to 'x'

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(3axy)$$

$$\frac{d}{dx}(x^3+y^3) = 3a\frac{d}{dx}(xy)$$

$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3) = 3a\left(x\frac{dy}{dx} + y\frac{d}{dx}x\right)$$

$$3x^2+3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx}+y\right)$$

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay-3x^2$$

$$\frac{dy}{dx}(3y^2-3ax) = 3ay-3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{3ay-3x^2}{3y^2-3ax}}$$

→ **Higher order derivatives :**

$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \text{------}$ are Higher order derivatives.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

(20) If $y=(x^2+4)^3$ then find $\frac{d^2y}{dx^2}$

(or)

Find second order derivative of the function $(x^2+4)^3$.

Sol:-

$$y=(x^2+4)^3$$

D.w.r.to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^2+4)^3$$

$$= 3(x^2+4)^{3-1}\cdot\frac{d}{dx}(x^2+4)$$

$$= 3(x^2+4)^2\left(\frac{d}{dx}x^2+\frac{d}{dx}4\right)$$

$$= 3(x^2+4)^2(2x+0)$$

$$\frac{dy}{dx} = 6x.(x^2+4)^2 = 6x(x^4+16+8x^2)$$

$$\frac{dy}{dx} = 6x^5+96x+48x^3$$

Again D.w.r.to 'x'

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(6x^5+96x+48x^3)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}6x^5+\frac{d}{dx}96x+\frac{d}{dx}48x^3$$

$$= 6\frac{d}{dx}x^5+96\frac{d}{dx}x+48\frac{d}{dx}x^3$$

$$= 6(5x^4)+96(1)+48(3x^2)$$

$$\frac{d^2y}{dx^2} = 30x^4+96+144x^2$$

Maxima and Minima

Increasing function :- If $y = f(x)$ is any function then 'y' is said to be an increasing function of 'x' at the point $x = a$ if $\left[\frac{dy}{dx} \text{ at } x = a\right] > 0 \Rightarrow \boxed{\left(\frac{dy}{dx}\right)_{at\ x=a} > 0}$

Decreasing function :- If $y = f(x)$ is any function then 'y' is said to be an decreasing function of 'x' at the point $x = a$ if $\boxed{\left(\frac{dy}{dx}\right)_{at\ x=a} < 0.}$

Def. of Maxima and Minima :-

→ A function $f(x)$ is said to have attained its "Maximum Value" at $x = a$ if the function ceases to increase and begins to decrease at $x = a$.

→ A function $f(x)$ is said to have attained its "(Minimum) minimum Value" at $x = b$ if the function ceases to decrease and begins to increase at $x = b$.

Necessary and sufficient condition for Maxima and Minima.

	Maxima	Minima
Necessary Condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Sufficient Condition	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} > 0$

(20) Find the Maximum and Minimum values of the function $2x^3-15x^2+36x+12$

Sol:- Let $y = 2x^3-15x^2+36x+12$.

The necessary condition to find turning points is $\boxed{\frac{dy}{dx}=0}$.

$y = 2x^3-15x^2+36x+12$

D.w.r.to 'x'

$\frac{dy}{dx} = \frac{d}{dx}(2x^3-15x^2+36x+12)$

$\frac{dy}{dx} = \frac{d}{dx}2x^3 - \frac{d}{dx}15x^2 + \frac{d}{dx}36x + \frac{d}{dx}12$

$\frac{dy}{dx} = 2\frac{d}{dx}x^3 - 15\frac{d}{dx}x^2 + 36\frac{d}{dx}x + \frac{d}{dx}12$

$\frac{dy}{dx} = 2(3x^2) - 15(2x) + 36(1) + 0$

$\boxed{\frac{dy}{dx} = 6x^2-30x+36} \Rightarrow \boxed{\frac{dy}{dx}=0}$

$6x^2-30x+36=0$

$6(x^2-5x+6)=0$

$x^2-5x+6=0$

$x^2-2x-3x+6=0$

$x(x-2)-3(x-2)=0$

$(x-2)(x-3)=0$

$x-2=0$ (or) $x-3=0$

$\boxed{x=2}$ (or) $\boxed{x=3}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(6x^2-30x+36)$

$= \frac{d}{dx}6x^2 - \frac{d}{dx}30x + \frac{d}{dx}36$

$= 6\frac{d}{dx}x^2 - 30\frac{d}{dx}x + \frac{d}{dx}36$

$$\frac{d^2y}{dx^2} = 6(2x) - 30(1) + 0 = 12x - 30$$

$$\boxed{\frac{d^2y}{dx^2} = 12x - 30}$$

At $x = 2 \Rightarrow \frac{d^2y}{dx^2} = 12(2) - 30 = 24 - 30 = -6 < 0$

$\therefore \frac{d^2y}{dx^2} < 0$ at $x = 2$

So the given function attains to Maxima

Maximum value of $y = 2x^3 - 15x^2 + 36x + 12$

At $x = 2$

$= 2(2)^3 - 15(2)^2 + 36(2) + 12$

$= 16 - 60 + 72 + 12$

$\boxed{\text{Maximum value of } y = 40.}$

At $x = 3 \Rightarrow \frac{d^2y}{dx^2} = 12x - 30 = 12(3) - 30 = 36 - 30 = 6 > 0$

$\therefore \frac{d^2y}{dx^2} > 0$ at $x = 3$.

So the given function attains to Minima

Minimum value of $y = 2x^3 - 15x^2 + 36x + 12$

At $x = 3$

$= 2(3)^3 - 15(3)^2 + 36(3) + 12$

$= 54 - 135 + 108 + 12$

$= 174 - 135$

$\boxed{\text{Minimum value of } y = 39}$

$\therefore$ Maximum value of $y = 40$

$\therefore$ Minimum value of $y = 39$.

21. A company has examined its cost structure and revenue structure and has determined that 'C' is the total cost, 'R' is the total Revenue, 'x' is the number of units produced are related as $C = 100 + 0.015x^2$ and $R = 3x$. Find the production rate 'x' that will maximise profits of the company. Find the profit. and also find profit when $x = 120$.

Sol:- Profit (P) = Revenue − Cost

$$P = R - C$$

$$P = (3x) - (100 + 0.015x^2)$$

$$\boxed{P = 3x - 100 - 0.015x^2} \longrightarrow (1)$$

For Max. & Min. Necessary condition is $\frac{dP}{dx} = 0$

$$P = 3x - 100 - 0.015x^2$$

D.w.r to 'x'

$$\frac{dP}{dx} = \frac{d}{dx}(3x - 100 - 0.015x^2)$$

$$= \frac{d}{dx}(3x) - \frac{d}{dx}100 - \frac{d}{dx}0.015x^2$$

$$= 3\frac{d}{dx}x - \frac{d}{dx}100 - 0.015\frac{d}{dx}x^2$$

$$= 3(1) - 0 - 0.015(2x)$$

$$\frac{dP}{dx} = 3 - 0.03x$$

For Maxima & Minima $\frac{dP}{dx} = 0$

$$3 - 0.03x = 0$$

$$0.03x = 3$$

$$\frac{3}{100}x = 3$$

$$x = \frac{300}{3} = 100$$

$\boxed{x = 100}$ units

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}(3-0.03x)$$

$$= \frac{d}{dx}(3) - \frac{d}{dx}(0.03x)$$

$$= 0 - 0.03\frac{d}{dx}(x)$$

$$\frac{d^2P}{dx^2} = -0.03(1) = -0.03 < 0$$

$$\therefore \frac{d^2P}{dx^2} < 0$$

∴ The profit function attains to maxima.

Maximum profit of $P = 3x - 100 - 0.015x^2$

At $x = 100$

$$= 3(100) - 100 - 0.015(100)^2$$

$$= 300 - 100 - 150$$

$$\boxed{\text{Maximum profit} = \text{Rs. } 50}$$

If $x = 120$ from ①

$$\text{Profit } (P) = 3x - 100 - 0.015x^2$$

$$= 3(120) - 100 - 0.015(120)^2$$

$$\boxed{\text{Profit} = \text{Rs } 44}$$

(22) Find two numbers whose sum is 24 and whose product is as large as possible (Maximum).

Sol:- Let the two numbers are x and y

sum $= 24$

$x + y = 24 \Rightarrow \boxed{y = 24 - x}$

According to problem product is maximum

Product $P = x \cdot y$

$$P = x(24 - x)$$

$$\boxed{P = 24x - x^2}$$

For Maxima & Minima Necessary condition is $\boxed{\frac{dP}{dx}=0}$

$$P = 24x - x^2$$

D.w.r to x

$$\frac{dP}{dx} = \frac{d}{dx}(24x - x^2)$$

$$= \frac{d}{dx}(24x) - \frac{d}{dx}(x^2)$$

$$= 24\frac{d}{dx}(x) - \frac{d}{dx}(x^2)$$

$$\boxed{\frac{dP}{dx} = 24 - 2x}$$

$$\therefore \frac{dP}{dx} = 0$$

$$24 - 2x = 0$$

$$2x = 24$$

$$x = \frac{24}{2} = 12$$

$$\boxed{x=12}$$

$$\therefore \frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}(24-2x) = \frac{d}{dx}(24) - \frac{d}{dx}(2x)$$

$$= 0 - 2 = -2 < 0$$

$$\therefore \boxed{\frac{d^2P}{dx^2} = -2 < 0}$$

The function attains to Maxima.

So $x=12$ then $y = 24 - x = 24 - 12 = 12$

$$\boxed{y=12}$$

$\therefore$ Two numbers are $x=12$ & $y=12$.

Matrices (Previous year question)

1. If $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$ Show that

$A(B+C) = AB + AC$.

Sol:- LHS $\Rightarrow$ $A(B+C)$

$$B+C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} (2\times3)+(4\times3) & (2\times1)+(4\times6) \\ (6\times3)+(8\times3) & (6\times1)+(8\times6) \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 18 & 26 \\ 42 & 54 \end{bmatrix} \rightarrow (1)$$

RHS $\Rightarrow$ $AB + AC$

$$AB = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (2\times1)+(4\times3) & (2\times2)+(4\times4) \\ (6\times1)+(8\times3) & (6\times2)+(8\times4) \end{bmatrix}$$

$$AB = \begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} (2\times2)+(4\times0) & (2\times-1)+(4\times2) \\ (6\times2)+(8\times0) & (6\times-1)+(8\times2) \end{bmatrix}$$

$$AC = \begin{bmatrix} 4 & 6 \\ 12 & 10 \end{bmatrix}$$

$$\therefore AB+AC = \begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 12 & 10 \end{bmatrix} = \begin{bmatrix} 18 & 26 \\ 42 & 54 \end{bmatrix} \rightarrow (2)$$

From (1) & (2)

$A(B+C) = AB + AC$.

Hence Proved