Matrices

① **Matrix :-** An ordered rectangular array of elements is called a matrix.

Ex :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{matrix} \rightarrow 1^{st} \text{ row} \\ \rightarrow 2^{nd} \text{ row} \end{matrix}$ $\quad B = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 5 & 6 \\ 6 & 2 & 3 \end{bmatrix}$

$\downarrow 1^{st}$ column $\quad \downarrow 2^{nd}$ column.

② **Order of Matrix :-** A matrix having 'm' rows and 'n' columns is said to be of order $m \times n$. (Read as m by n)

Ex :- $A = \begin{bmatrix} 7 & 2 \\ 3 & 9 \end{bmatrix}_{2\times 2}$ order $= 2\times 2$.

$B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$ order $= 2\times 3$.

③ **Types of Matrices :**

(i) **Square Matrix :-** A matrix in which the number of rows is equal to the number of columns is called a square matrix.

Ex :- $A = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}_{2\times 2}$ $\quad B = \begin{bmatrix} 5 & 2 & -3 \\ -4 & 6 & 7 \\ 1 & 0 & 2 \end{bmatrix}_{3\times 3}$

$C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Here a_{11}, a_{22}, a_{33} are principal diagonal elements.

(ii) **Rectangular Matrix :-** A Matrix in which the number of Rows not equal to number of columns is called a Rectangular matrix.

Ex :- $A = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 3 & 9 \end{bmatrix}_{2\times 3}$

(iii) **Row matrix :-** A matrix having single row is called a Row matrix.

Ex :- $A = \begin{bmatrix} 1 & 5 & 6 \end{bmatrix}_{1\times 3}$

(iv) Column Matrix :- A matrix having only one column is called a column matrix.

Ex :- $A = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}_{3\times 1}$

(v) Zero (Null) Matrix :- If each element of a matrix is zero then it is called a Null matrix.

Ex :- $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(vi) Diagonal Matrix :- If each non-diagonal element of a square matrix is equal to zero then the matrix is called a diagonal matrix.

Ex :- $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \text{diag.}(2 \;\; 4 \;\; 6)$

(vii) Scalar Matrix :- If each non diagonal element of a square matrix is zero and all diagonal elements are equal. ~~to each other~~ then it is called a Scalar Matrix.

Ex :- $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

(viii) Unit (Identity) Matrix :- If each non diagonal element of a square matrix is equal to zero and each diagonal element equal to 1. then the matrix is called a Unit (Identity) Matrix.

Ex :- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

(ix) Triangular Matrices :-

A square matrix $A = [a_{ij}]$ is said to be "Upper Triangular" if $a_{ij} = 0$ for all $i > j$.

A square matrix $A = [a_{ij}]$ is said to be "Lower Triangular" if $a_{ij} = 0$ for all $i < j$.

Ex :-
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$ Upper triangular matrix.

$$B = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$ Lower triangular matrix.

(x) Transpose of a matrix :-

If $A = [a_{ij}]$ is an $m \times n$ matrix then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted by A^T (or) A'

Ex :- If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

(xi) Symmetric matrix :-

A square matrix 'A' is said to be symmetric if $\boxed{A = A^T}$

(xii) Skew - symmetric matrix :-

A square matrix 'A' is said to be skew - symmetric if $\boxed{A = -A^T}$

(xiii) Trace of a matrix :- The sum of the elements of the principal diagonal of a square matrix 'A' is called the Trace of A and it is denoted by Tr(A).

Ex :- $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \\ 7 & 0 & 2 \end{bmatrix}$ then $Tr(A) = 1+5+2 = 8$.

→ Properties of transpose of matrices :-

A, B are any two matrices then

(i) $(A^T)^T = A$ (ii) $(A+B)^T = A^T + B^T$ (iii) $(KA^T) = KA^T$ (iv) $(AB)^T = B^T \cdot A^T$

→ Equality of matrices :- Matrices A and B are said to be equal if A and B are of the same order and the corresponding elements of A and B are the same.

→ Addition (sum) of two matrices :- A and B are matrices of the same order. then the sum of A and B is denoted by A+B is defined as the matrix of the same order in which each element is the sum of the corresponding elements of A and B.

Properties of addition of matrices :-

(a) Commutative property : $A+B = B+A$

(b) Associative " : $A+(B+C) = (A+B)+C$.

(c) Additive Identity : $A+O = O+A = A$.

(d) Additive Inverse : $A+B = O = B+A$. (Here $B = -A$)

(O is the zero matrix)

Scalar multiplication of a matrix :- Let A be a matrix of order $m \times n$ and 'K' be a scalar then the matrix obtained by multiplying each element of 'A' by K is called a scalar multiplication of 'A' and it is denoted by KA.

Ex:- If $A = \begin{bmatrix} 1 & 3 & 6 \\ 4 & 2 & 5 \end{bmatrix}$ and 'K = 2' then

$$KA = 2A = 2\begin{bmatrix} 1 & 3 & 6 \\ 4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 12 \\ 8 & 4 & 10 \end{bmatrix}$$

Multiplication of matrices :- Two matrices A and B are conformable for multiplication in that order (product of A B) if the number of columns of A is equal to the number of rows of B.

Properties of multiplication of matrices

(i) Associative law :- $(AB)C = A(BC)$

(ii) Multiplicative Identity :- $AI = A = IA$

(iii) The distributive law :- (a) $A(B+C) = AB + AC$ (Left Distributive Law)
(b) $(A+B)C = AC + BC$ (Right Distributive Law)

Problems.

① If $A = \begin{bmatrix} 2 & 3 & -1 \\ 7 & 8 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & -1 \end{bmatrix}$ then find A+B.

Sol:-

$$A + B = \begin{bmatrix} 2 & 3 & -1 \\ 7 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 2+1 & 3+0 & -1+1 \\ 7+2 & 8-4 & 5-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 9 & 4 & 4 \end{bmatrix}$$

② If $\begin{bmatrix} x-1 & 2 & y-5 \\ z & 0 & 2 \\ 1 & -1 & 1+a \end{bmatrix} = \begin{bmatrix} 1-x & 2 & -y \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ then find the values of x, y, z and a.

Sol:- Two matrices are equal then their corresponding elements are equal.

$x-1=1-x$, $2x=2$, $\boxed{x=1}$

$y-5=-y$, $2y=5$, $\boxed{y=\frac{5}{2}}$

$z=2$,

$1+a=1$, $\boxed{a=0}$

③ If $A=\begin{bmatrix}-1 & -2 & 3\\ 1 & 2 & 4\\ 2 & -1 & 3\end{bmatrix}$ then find $5A$.

Sol:-

$$5A = 5\begin{bmatrix}-1 & -2 & 3\\ 1 & 2 & 4\\ 2 & -1 & 3\end{bmatrix} = \begin{bmatrix}-5 & -10 & 15\\ 5 & 10 & 20\\ 10 & -5 & 15\end{bmatrix}$$

④ If $A=\begin{bmatrix}3 & 2 & -1\\ 2 & -2 & 0\\ 1 & 3 & 1\end{bmatrix}$, $B=\begin{bmatrix}-3 & -1 & 0\\ 2 & 1 & 3\\ 4 & -1 & 2\end{bmatrix}$ and $X=A+B$ then find 'X'.

Sol:-

$$X = \begin{bmatrix}3 & 2 & -1\\ 2 & -2 & 0\\ 1 & 3 & 1\end{bmatrix} + \begin{bmatrix}-3 & -1 & 0\\ 2 & 1 & 3\\ 4 & -1 & 2\end{bmatrix} = \begin{bmatrix}0 & 1 & -1\\ 4 & -1 & 3\\ 5 & 2 & 3\end{bmatrix}$$

⑤ If $A=\begin{bmatrix}0 & 1 & 2\\ 2 & 3 & 4\\ 4 & 5 & -6\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$ find $4A-5B$.

Sol:-

$$4A-5B = 4\begin{bmatrix}0 & 1 & 2\\ 2 & 3 & 4\\ 4 & 5 & -6\end{bmatrix} - 5\begin{bmatrix}-1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 4 & 8\\ 8 & 12 & 16\\ 16 & 20 & -24\end{bmatrix} - \begin{bmatrix}-5 & 10 & 15\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix}$$

$$= \begin{bmatrix}5 & -6 & -7\\ 8 & 7 & 16\\ 16 & 20 & -19\end{bmatrix}$$

⑥ Find the trace of $A=\begin{bmatrix}1 & 3 & -5\\ 2 & -1 & 5\\ 2 & 0 & 1\end{bmatrix}$

Sol:-

Trace of $A = 1+(-1)+1 = 1.$

7) Construct a 3×2 matrix whose elements are defined by $a_{ij} = \frac{1}{2}|i-3j|$.

Sol:- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times2}$ & $a_{ij} = \frac{1}{2}|i-3j|$, $i=1,2,3$ and $j=1,2$.

$a_{11} = \frac{1}{2}|1-(3\times1)| = 1$, $a_{12} = \frac{1}{2}|1-(3\times2)| = \frac{5}{2}$

$a_{21} = \frac{1}{2}|2-(3\times1)| = \frac{1}{2}$, $a_{22} = \frac{1}{2}|2-(3\times2)| = 2$

$a_{31} = \frac{1}{2}|3-(3\times1)| = 0$, $a_{32} = \frac{1}{2}|3-(3\times2)| = \frac{3}{2}$

$\therefore A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$

8) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$ and $2X + A = B$ then find X.

Sol:- Given $2X + A = B \Rightarrow 2X = B - A$

$X = \frac{1}{2}(B-A)$

$X = \frac{1}{2}\left(\begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix} 3-1 & 8-2 \\ 7-3 & 2-4 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{2} & \frac{6}{2} \\ \frac{4}{2} & \frac{-2}{2} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

$\therefore X = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$.

9) If $A = \begin{bmatrix} 3 & -2 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix}$ then find AB.

Sol:- $AB = \begin{bmatrix} 3 & -2 \\ 1 & 6 \end{bmatrix}_{2\times2} \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix}_{2\times2} = \begin{bmatrix} (3\times4)+(-2\times2) & (3\times-1)+(-2\times5) \\ (1\times4)+(6\times2) & (1\times-1)+(6\times5) \end{bmatrix}$

$= \begin{bmatrix} 8 & -13 \\ 16 & 29 \end{bmatrix}$

(10) If $A = \begin{bmatrix} -1 & 4 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$ then find AB.

Sol:- $AB = \begin{bmatrix} -1 & 4 & 2 \end{bmatrix}_{1\times3} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}_{3\times1} = \begin{bmatrix} (-1\times5) + (4\times1) + (2\times3) \end{bmatrix}_{1\times1} = \begin{bmatrix} 5 \end{bmatrix}_{1\times1}$

In A No. of columns = 3

In B No. of Rows = 3

(11) If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -2 & -3 & 4 \\ 2 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix}$ find AB.

Sol:- $AB = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}_{3\times3} \begin{bmatrix} -2 & -3 & 4 \\ 2 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix}_{3\times3}$

No. of columns in A = 3

No. of Rows in B = 3

$\therefore$ AB exist.

$$AB = \begin{bmatrix} (2\times-2)+(2\times2)+(1\times1) & (2\times-3)+(2\times2)+(1\times2) & (2\times4)+(2\times-3)+(1\times-2) \\ (1\times-2)+(0\times2)+(2\times1) & (1\times-3)+(0\times2)+(2\times2) & (1\times4)+(0\times-3)+(2\times-2) \\ (2\times-2)+(1\times2)+(2\times1) & (2\times-3)+(1\times2)+(2\times2) & (2\times4)+(1\times-3)+(2\times-2) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(12) If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ Show that $A^3 - 3A^2 - A + 9I = 0$.

Sol:- $A^2 = A\cdot A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix}$

$A^3 = A^2\cdot A = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix}$

$\therefore A^3 - 3A^2 - A + 9I = \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix} - 3\begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix} - \begin{bmatrix} 12 & 9 & 0 \\ -9 & 6 & -6 \\ 18 & 12 & 15 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(13) If $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 3 & -5 \end{bmatrix}$ show that $(A+B)^T = A^T + B^T$

Sol:- LHS $\Rightarrow (A+B)^T$

$$A+B = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 4 \\ 1 & 3 & -5 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 5 \\ 5 & 5 & -2 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} 5 & 5 \\ -5 & 5 \\ 5 & -2 \end{bmatrix} \rightarrow (1)$$

RHS $\Rightarrow A^T + B^T$

$$\therefore A^T + B^T = \begin{bmatrix} 2 & 4 \\ -3 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ -2 & 3 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -5 & 5 \\ 5 & -2 \end{bmatrix} \rightarrow (2)$$

From (1) & (2)

$$(A+B)^T = A^T + B^T$$

(14) If $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$ then verify that $(AB)^T = B^T A^T$

Sol:- LHS $\Rightarrow (AB)^T$ So $AB = \underset{2\times 3}{\begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}} = \begin{bmatrix} 15 & 4 \\ -28 & -18 \end{bmatrix}$

$$\therefore (AB)^T = \begin{bmatrix} 15 & -28 \\ 4 & -18 \end{bmatrix} \rightarrow (1)$$

RHS $\Rightarrow$

$$B^T A^T = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 15 & -28 \\ 4 & -18 \end{bmatrix} \rightarrow (2)$$

From (1) & (2) $(AB)^T = B^T . A^T$.

Determinants.

→ The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad-bc$.

→ **Minor of an element** :- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$.

The minor of an element in this matrix is the determinant of 2×2 matrix obtained after deleting the row and the column in which the element is present.

Ex:- The minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$

The minor of $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}a_{23} - a_{13}a_{21}$

→ **Co factor of an element** :- If we multiply the minor of the element in the i^{th} row and j^{th} column of the determinant of the matrix by $(-1)^{i+j}$ the product is called the co-factor of the element.

∴ co-factor of a_{ij} is $A_{ij} = (-1)^{i+j} \times$ minor of a_{ij}

→ Determinant of a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the sum of the products of elements of the first row with their corresponding cofactors.

It is denoted by $\det A$ (OR) $|A|$.

NOTE:- We can expand with any row or column.

Properties of determinants

① If any two rows (columns) of a determinant are identical the value of the determinant is zero.

② If any two rows (columns) are interchanged the value of the determinant is the negative of the value of the original determinant.

③ If the rows of a determinant are changed into columns and vice versa, the value of the determinant remains unchanged.

④ If the elements of a row (column) of a matrix are multiplied by the number 'K' the determinant of the matrix thus obtained is 'K' times the determinant of the original matrix.

⑤ If the elements of any row & column of a determinant is sum (difference) of two & more elements then the determinant can be expressed as sum (difference) of two & more determinants.

Problems

① Find the determinant of $A = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$

Sol:- $|A| = \begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} = (1\times1) - (-1\times-3) = 1-3 = -2.$

② Find the minors of -1 and 3 in the matrix $A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{bmatrix}$

Sol:- Minor of $-1 = \begin{vmatrix} 0 & 5 \\ -3 & 3 \end{vmatrix} = (0\times3) - (5\times-3) = 15$

Minor of $3 = \begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} = (2\times-2) - (0\times-1) = -4$

③ Find the co-factors of ~~matrix~~ elements 6, −9 in $A = \begin{bmatrix} 3 & 4 & 7 \\ -2 & 5 & 6 \\ 7 & 3 & -9 \end{bmatrix}$

Sol:- Element position ~~coefficient~~ of 6 ⇒ $\boxed{a_{23} = 6.}$

∴ Co-factor of 6 is $A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 4 \\ 7 & 3 \end{vmatrix} = (-1)^5 (9-28) = 19$

Co-factor of −9 is $A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix} = (-1)^6 (15+8) = 23.$

(Here $a_{33} = -9$)

NOTE:- ① $\det(AB) = (\det A) \cdot (\det B)$

② $\det(A^T) = \det A.$

③ $\det(A^n) = (\det A)^n$ (where n is order of a matrix)

Singular matrix:- A square matrix is said to be singular if its determinant is zero ($|A| = 0$)

Non-singular matrix:- A square matrix is said to be Non-singular if its determinant is not equal to zero ($|A| \neq 0$)

Adjoint of a matrix:- The Transpose of a cofactor matrix is called Adjoint of a matrix. it is denoted by Adj A.

Inverse of a matrix:- Let A is $n \times n$ matrix then there exist a matrix B such that $AB = BA = I$.

Here B is called inverse of A so $\boxed{B = A^{-1}}$

Here I is called Identity matrix.

→ If A is an $n \times n$ matrix then $\boxed{A^{-1} = \frac{1}{|A|} \text{Adj } A}$

Inverse exist if A is non-singular.

① If $A = \begin{bmatrix} -1 & -5 \\ -2 & 3 \end{bmatrix}$ then find A^{-1}.

Sol:-

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$|A| = (-1 \times 3) - (-5 \times -2) = -3 - 10 = -13.$$

∴ A is Non-singular matrix. A^{-1} exist.

$$\text{Adj } A = \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-13} \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{-3}{13} & \frac{-5}{13} \\ \frac{-2}{13} & \frac{1}{13} \end{bmatrix}$$

② Find the Inverse of a matrix $A = \begin{bmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{bmatrix}$

Sol:-

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$|A| = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{vmatrix} \end{array}$$

Expand with 1st Row.

$$= +1 \begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} - 0 \begin{vmatrix} -2 & 5 \\ 3 & 2 \end{vmatrix} - 4 \begin{vmatrix} -2 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= +1(4+5) - 0(-4-20) - 4(2-6)$$

$$= 9 - 0 + 16$$

$$\boxed{\det A = |A| = 25}$$

∴ A is non-singular matrix.

∴ A^{-1} exist.

Co-factor of $1 = (-1)^{1+1}$ Minor of $1 = (-1)^2 \begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} = 9$

" " " $0 = (-1)^{1+2}$ Minor of $0 = (-1)^3 \begin{vmatrix} -2 & 5 \\ 3 & 2 \end{vmatrix} = 19$

" " " $-4 = (-1)^{1+3}$ Minor of $-4 = (-1)^4 \begin{vmatrix} -2 & 2 \\ 3 & -1 \end{vmatrix} = -4$

" " " $-2 = (-1)^{2+1}$ Minor of $-2 = (-1)^3 \begin{vmatrix} 0 & -4 \\ -1 & 2 \end{vmatrix} = 4$

" " " $2 = (-1)^{2+2}$ Minor of $2 = (-1)^4 \begin{vmatrix} 1 & -4 \\ 3 & 2 \end{vmatrix} = 14$

" " " $5 = (-1)^{2+3}$ Minor of $5 = (-1)^5 \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} = 1$

" " " $3 = (-1)^{3+1}$ Minor of $3 = (-1)^4 \begin{vmatrix} 0 & -4 \\ 2 & 5 \end{vmatrix} = 8$

" " " $-1 = (-1)^{3+2}$ Minor of $-1 = (-1)^5 \begin{vmatrix} 1 & -4 \\ -2 & 5 \end{vmatrix} = 3$

" " " $2 = (-1)^{3+3}$ Minor of $2 = (-1)^6 \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} = 2$

Co-factor matrix of $A = \begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$

Adj. A = Transpose of a co-factor matrix of A.

$= \begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}^T$

Adj A $= \begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}$ Adj $A = \frac{1}{25} \begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{9}{25} & \frac{4}{25} & \frac{8}{25} \\ \frac{19}{25} & \frac{14}{25} & \frac{3}{25} \\ \frac{-4}{25} & \frac{1}{25} & \frac{2}{25} \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} \frac{9}{25} & \frac{4}{25} & \frac{8}{25} \\ \frac{19}{25} & \frac{14}{25} & \frac{3}{25} \\ \frac{-4}{25} & \frac{1}{25} & \frac{2}{25} \end{bmatrix}$

⇒ system of linear equations can be expressed in the form $\boxed{AX=D}$

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3\,.$$

Coefficient matrix $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Variable matrix $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Constant matrix $D=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

$\therefore AX=D \Rightarrow \boxed{X=A^{-1}\cdot D}$

Cramer's Rule

The system of linear equations $a_1x+b_1y+c_1z=d_1$
$a_2x+b_2y+c_2z=d_2$
$a_3x+b_3y+c_3z=d_3\,.$

$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

$\Delta_1=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_2=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$\Delta=|A|=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\therefore x=\dfrac{\Delta_1}{\Delta}$, $y=\dfrac{\Delta_2}{\Delta}$, $z=\dfrac{\Delta_3}{\Delta}$.

① Solve the following simultaneous linear equations by using Cramer's Rule.

$$3x+4y+5z=18.$$
$$2x-y+8z=13$$
$$5x-2y+7z=20.$$

Sol:-

$$A=\begin{bmatrix}3&4&5\\2&-1&8\\5&-2&7\end{bmatrix},\quad x=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad D=\begin{bmatrix}18\\13\\20\end{bmatrix}$$

$$\det A=\Delta=\begin{array}{c}+\quad\;-\quad\;+\\ \begin{vmatrix}3&4&5\\2&-1&8\\5&-2&7\end{vmatrix}\end{array}=3\begin{vmatrix}-1&8\\-2&7\end{vmatrix}-4\begin{vmatrix}2&8\\5&7\end{vmatrix}+5\begin{vmatrix}2&-1\\5&-2\end{vmatrix}$$

$$=3(-7+16)-4(14-40)+5(-4+5)$$
$$=27+104+5$$
$$=136$$

$$\Delta_1=\begin{array}{c}+\quad\;-\quad\;+\\ \begin{vmatrix}18&4&5\\13&-1&8\\20&-2&7\end{vmatrix}\end{array}=18\begin{vmatrix}-1&8\\-2&7\end{vmatrix}-4\begin{vmatrix}13&8\\20&7\end{vmatrix}+5\begin{vmatrix}13&-1\\20&-2\end{vmatrix}$$

$$=18(-7+16)-4(91-160)+5(-26+20)$$
$$=18(9)-4(-69)+5(-6)$$
$$=162+276-30$$
$$\boxed{\Delta_1=408}$$

$$\Delta_2=\begin{array}{c}+\quad\;-\quad\;+\\ \begin{vmatrix}3&18&5\\2&13&8\\5&20&7\end{vmatrix}\end{array}=3\begin{vmatrix}13&8\\20&7\end{vmatrix}-18\begin{vmatrix}2&8\\5&7\end{vmatrix}+5\begin{vmatrix}2&13\\5&20\end{vmatrix}$$

$$=3(91-160)-18(14-40)+5(40-65)$$
$$=-207+468-125$$
$$\boxed{\Delta_2=136}$$

$$\Delta_3=\begin{array}{c}+\quad\;-\quad\;+\\ \begin{vmatrix}3&4&18\\2&-1&13\\5&-2&20\end{vmatrix}\end{array}=3\begin{vmatrix}-1&13\\-2&20\end{vmatrix}-4\begin{vmatrix}2&13\\5&20\end{vmatrix}+18\begin{vmatrix}2&-1\\5&-2\end{vmatrix}$$

$$=3(-20+26)-4(40-65)+18(-4+5)$$
$$=18+100+18$$
$$\boxed{\Delta_3=136}$$

$$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{408}{136} = 3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{136}{136} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{136}{136} = 1$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$