Centered Differences :- The Newton forward and backward difference formulas are not appropriate for approximating $f(x)$ when x lies near the center of the table because neither will permit the highest-order difference to have x_0 close to x. A number of divided-difference formulas are available for this case each of which has situations when it can be used to maximum advantage these methods are known as "centered-difference formulae". We will consider only one centered difference formula. "Stirling's method"

The "Stiriling's formula" is

$$P_n(x) = P_{2m+1}(x) = f[x_0] + \frac{sh}{2}\left(f[x_{-1}, x_0] + f[x_0, x_1]\right) + s^2h^2 f[x_{-1}, x_0, x_1]$$
$$+ \frac{s(s^2-1)h^3}{2}\left(f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]\right)$$
$$+ \cdots + s^2(s^2-1)(s^2-4)\cdots\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \cdots x_m]$$
$$+ \frac{s(s^2-1)\cdots(s^2-m^2)h^{2m+1}}{2}\left(f[x_{-m-1}, \cdots x_m] + f[x_{-m}, \cdots x_{m+1}]\right)$$

Table :-

x	$f(x)$	First divided difference	Second d.d	Third d.d	Fourth d.d
x_{-2}	$f[x_{-2}]$				
		$f[x_{-2}, x_{-1}]$			
x_{-1}	$f[x_{-1}]$		$f[x_{-2}, x_{-1}, x_0]$		
		$f[x_{-1}, x_0]$		$f[x_{-2}, x_{-1}, x_0, x_1]$	
x_0	$f[x_0]$		$f[x_{-1}, x_0, x_1]$		$f[x_{-2}, x_{-1}, x_0, x_1, x_2]$
		$f[x_0, x_1]$		$f[x_{-1}, x_0, x_1, x_2]$	
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$			
x_2	$f[x_2]$				

Exercise-3.3.

9 (c) Approximate $f(0.43)$ by strilings formula using the following data:

x	0.0	0.2	0.4	0.6	0.8
$f(x)$	1.00000	1.22140	1.49182	1.82212	2.22554.

Sol:-

	x	$f(x)$	1st divided diff.	2nd d.d	3rd d.d	4th d.d
x_{-2}	0	1 $f(x_{-2})$				
			$f[x_{-2},x_{-1}]$ 1.107			
				$f[x_{-2},x_{-1},x_0]$ 0.6128.		
					$f[x_{-2},x_{-1},x_0,x_1]$ 0.2262.	
x_{-1}	0.2	1.22140 $f(x_{-1})$				
			1.3521 $f[x_{-1},x_0]$			
x_0	0.4	1.49182 $f(x_0)$		0.7485 $f[x_{-1},x_0,x_1]$		0.0620. $f[x_{-2},x_{-1},x_0,x_1,x_2]$
			1.6515 $f[x_0,x_1]$		0.2758 $f[x_{-1},x_0,x_1,x_2]$	
x_1	0.6	1.82212 $f(x_1)$		0.914 $f[x_0,x_1,x_2]$		
			2.0171 $f[x_1,x_2]$			
x_2	0.8	2.22554 $f(x_2)$				

By the striling's formula

$$P_n(x) = f[x_0] + \frac{sh}{2}\left(f[x_{-1},x_0] + f[x_0,x_1]\right) + s^2h^2 f[x_{-1},x_0,x_1] + \frac{s(s^2-1)h^3}{2}$$

$$\left(f[x_{-2},x_{-1},x_0,x_1] + f[x_{-1},x_0,x_1,x_2]\right) + s^2(s^2-1)h^4\left(f[x_{-2},x_{-1},x_0,x_1,x_2]\right)$$

$+$ ———

$\therefore\ x = 0.43,$ $\boxed{h=0.2.}$ $\quad S = \frac{x-x_0}{h} = \frac{0.43-0.4}{0.2} = 0.15$ $\quad \boxed{S=0.15}$

$$P(0.43) = 1.49182 + \frac{(0.15)(0.2)}{2}(1.3521+1.6515) + (0.15)^2(0.2)^2(0.7485)$$

$$+ \frac{(0.15)\left((0.15)^2-1\right)(0.2)^3}{2}(0.2262+0.2758)$$

$$+ (0.15)^2\left((0.15)^2-1\right)(0.2)^4(0.0620).$$

$$\boxed{P(0.43) = f(0.43) = 1.53725}$$

(7) Using Newton's divided difference formula construct the interpolating polynomial of degree three for the unequal spaced points given in the following table.

X	−0.1	0	0.2	0.3
f(X)	5.30000	2.00000	3.19000	1.00000

Sol:-

	X	f(X)	1st divide diff.	2nd divide diff	3rd divide diff.
x_0	−0.1	$f(x_0)$ 5.30000			
			$f(x_0,x_1)$ −33		
				$f(x_0,x_1,x_2)$ 129.833	
x_1	0	$f(x_1)$ 2.00000			$f(x_0,x_1,x_2,x_3)$ −556.6665
			$f(x_1,x_2)$ 5.95		
				$f(x_1,x_2,x_3)$ −92.8333	
x_2	0.2	$f(x_2)$ 3.19000			
			$f(x_2,x_3)$ −21.9		
x_3	0.3	$f(x_3)$ 1.00000			

$$f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0}, \quad f[x_0,x_1,x_2] = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}.$$

$$f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}, \quad f[x_0,x_1,x_2,x_3] = \frac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{x_3-x_0}$$

—— like that.

Newton's divide difference formula.

$$P_3(x) = f(x_0) + f(x_0,x_1)(x-x_0) + f(x_0,x_1,x_2)(x-x_0)(x-x_1) + f(x_0,x_1,x_2,x_3)(x-x_0)(x-x_1)(x-x_2).$$

$$P_3(x) = 5.30000 + (-33)(x-(-0.1)) + 129.833\,(x+0.1)(x-0) + (-556.6665)(x+0.1)(x-0)(x-0.2).$$

$$P_3(x) = 5.3 - 33(x+0.1) + 129.833\,x(x+0.1) - 556.6665\,x(x+0.1)(x-0.2).$$

3.4: Hermite Interpolation

Let $x_0, x_1, \ldots x_n$ be $n+1$ distinct numbers in $[a,b]$ and for $i = 0,1,\ldots n$. let m_i be a non-negative integer suppose that $f \in C^m[a,b]$, where $m = \max. m_i$ $(0 \le i \le n)$

The osculating polynomial approximating f is the polynomial $P(x)$ of least degree. such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}, \quad \text{for each } i = 0,1,\ldots n \quad k = 0,1,\ldots m_i.$$

when $m_i = 1$ for each $i = 0,1,\ldots n$ gives the Hermite polynomials.

Procedure of Hermite Interpolation

① To obtain the coefficients of the Hermite interpolating polynomial $H(x)$ on the $(n+1)$ distinct numbers $x_0, x_1, \ldots x_n$ for the function f.

② Give the numbers $x_0, x_1, \ldots x_n$ values $f(x_0), f(x_1) \ldots f(x_n)$ and $f'(x_0), f'(x_1), \ldots f'(x_n)$.

③ let $z_{2i} = x_i$ | $Q_{2i,0} = f(x_i)$ | $Q_{2i+1,1} = f'(x_i)$.

$z_{2i+1} = x_i$ | $Q_{2i+1,0} = f(x_i)$

for $i = 0,1,2,\ldots n$

④

$$Q_{i,j} = \frac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}} \quad \text{for } i = 2,3,\ldots 2n+1 \quad j = 2,3,\ldots i$$

5 The Hermite interpolating polynomial is

$$H(x) = Q_{0,0} + Q_{1,1}(x-x_0) + Q_{2,2}(x-x_0)^2 + Q_{3,3}(x-x_0)^2(x-x_1)$$

$$+ Q_{4,4}(x-x_0)^2(x-x_1)^2 + \cdots + Q_{2n+1,2n+1}(x-x_0)^2(x-x_1)^2 \cdots (x-x_{n-1})^2(x-x_n).$$

Table :-

z_i	$Q_{i,0}$	$Q_{i,1}$	$Q_{i,2}$	------	$Q_{i,2n+1}$
z_0	$Q_{0,0}$				
z_1	$Q_{1,0}$	$Q_{1,1}$			
			$Q_{2,2}$		
					$Q_{2n+1,2n+1}$
z_2	$Q_{2,0}$	$Q_{2,1}$			
z_3	$Q_{3,0}$	$Q_{3,1}$	$Q_{3,2}$		
z_{2n}	$Q_{2n,0}$				
z_{2n+1}	$Q_{2n+1,0}$	$Q_{2n+1,1}$			

Exercise-3.4

1 Construct an approximate polynomial by using Hermite interpolation from the following data:

x	$f(x)$	$f'(x)$
0.8	0.22363362	2.1691753
1	0.65809197	2.0466965

Sol:- Given $x_0 = 0.8 \quad x_1 = 1$

$f(x_0) = 0.22363362, \; f(x_1) = 0.65809197$

$f'(x_0) = 2.1691753, \; f'(x_1) = 2.0466965.$

(Here $n=1$, so it has $n+1 = 1+1 = 2$ points) those are x_0, x_1.

By Hermite interpolation $Q_{i,j} = \dfrac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$

where $i = 2, 3, \ldots 2n+1$

$j = 2, 3, \ldots i$.

where $x_i = z_{2i}$ and $x_i = z_{2i+1}$

$f(x_i) = Q_{2i,0}$ and $f(x_i) = Q_{2i+1,0}$

$f'(x_i) = Q_{2i+1,1}$

For $i=0 \Rightarrow x_0 = z_0 = z_1 = 0.8$

For $i=1 \Rightarrow x_1 = z_2 = z_3 = 1$

For $i=0 \Rightarrow Q_{0,0} = Q_{1,0} = f(x_0) = 0.22363362$

For $i=1 \Rightarrow Q_{2,0} = Q_{3,0} = f(x_1) = 0.65809197$

For $i=0 \Rightarrow Q_{1,1} = f'(x_0) = 2.1691753$

For $i=1 \Rightarrow Q_{3,1} = f'(x_1) = 2.0466965$.

$Q_{2,1} = \dfrac{Q_{2,0} - Q_{1,0}}{z_2 - z_1} = 2.172291$, $Q_{3,2} = \dfrac{Q_{3,1} - Q_{2,1}}{z_3 - z_1} = -0.615$

$Q_{2,2} = \dfrac{Q_{2,1} - Q_{1,1}}{z_2 - z_0} = 0.015579$, $Q_{3,3} = \dfrac{Q_{3,2} - Q_{2,2}}{z_3 - z_0} = -3.2177925$

z	$f(z)$	1st divide diff.	2nd d.d	3rd d.d
$x_0 = z_0 = 0.8$	$Q_{0,0}$			
		$Q_{1,1}$		
$x_0 = z_1 = 0.8$	$Q_{1,0}$		$Q_{2,2}$	
		$Q_{2,1}$		$Q_{3,3}$
$x_1 = z_2 = 1$	$Q_{2,0}$		$Q_{3,2}$	
		$Q_{3,1}$		
$x_1 = z_3 = 1$	$Q_{3,0}$			

By Hermite Interpolating Polynomial.

$$H_3(x) = Q_{0,0} + Q_{1,1}(x-x_0) + Q_{2,2}(x-x_0)^2 + Q_{3,3}(x-x_0)^2(x-x_1)$$

$$H_3(x) = 0.22363362 + 2.1691753(x-0.8) + 0.01558225(x-0.8)^2 - 3.2177925(x-0.8)^2(x-1).$$

3 (b) Using the Hermite Polynomial from above problem to approximate the value of x in $f(x) = \sin(e^x - 2)$ approximate $f(0.9)$. Find the absolute error. (Above problem data)

Sol:- $f(x) = \sin(e^x - 2)$

$f'(x) = \cos(e^x-2)(e^x) = e^x \cdot \cos(e^x - 2)$

$f(x_0) = f(0.8) = \sin(e^{0.8} - 2) = 0.2236336$

$f(x_1) = f(1) = \sin(e^1 - 2) = 0.65809196$

$f'(x_0) = f'(0.8) = e^{0.8}\cos(e^{0.8} - 2) = 2.16917527$

$f'(x_1) = f'(1) = e^1 \cos(e^1 - 2) = 2.04669647$

Hermite interpolating Polynomial is (from above problem)

$$H_3(x) = 0.22363362 + 2.1691753(x-0.8) + 0.01558225(x-0.8)^2 - 3.2177925(x-0.8)^2(x-1)$$

For $x = 0.9$ (substitute $x = 0.9$)

$$\boxed{H_3(0.9) = 0.443924765}$$

Given $f(x) = \sin(e^x - 2)$

$f(0.9) = \sin(e^{0.9} - 2)$

$f(0.9) = 0.44359243$ is the actual error of $f(x)$.

Absolute error $= |f(0.9) - H_3(0.9)|$

$= |0.44359243 - 0.443924765|$

$= 0.000332326.$

2 (a) Construct an approximating polynomial for the data using ~~[illegible]~~ Hermite Interpolation.

x	$f(x)$	$f'(x)$
0	1	2
0.5	2.71828	5.43656

Sol: By Hermite Interpolation $Q_{i,j} = \dfrac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$

where $i = 2, 3, \ldots 2n+1$
$j = 2, 3, \ldots i$

where $x_i = z_{2i}$ and $x_i = z_{2i+1}$

$f(x_i) = Q_{2i,0}$ and $f(x_i) = Q_{2i+1,0}$

$f'(x_i) = Q_{2i+1,1}$

Given $x_0 = 0$, $x_1 = 0.5$, $f(x_0) = 1$, $f(x_1) = 2.71828$

$f'(x_0) = 2$, $f'(x_1) = 5.43656.$

$x_0 = z_0 = z_1 = 0$ and $x_1 = z_2 = z_3 = 0.5$

$f(x_0) = Q_{0,0} = Q_{1,0} = 1$, $f'(x_0) = Q_{1,1} = 2$

$f(x_1) = Q_{2,0} = Q_{3,0} = 2.71828$, $f'(x_1) = Q_{3,1} = 5.43656.$

$$Q_{2,1} = \frac{Q_{2,0} - Q_{1,0}}{z_2 - z_1} = \frac{2.71828 - 1}{0.5 - 0} = 3.43656$$

$$Q_{2,2} = \frac{Q_{2,1} - Q_{1,1}}{z_2 - z_0} = \frac{3.43656 - 2}{0.5 - 0} = 2.87312$$

$$Q_{3,2} = \frac{Q_{3,1} - Q_{2,1}}{z_3 - z_1} = \frac{5.43656 - 3.43656}{0.5 - 0} = 4$$

$$Q_{3,3} = \frac{Q_{3,2} - Q_{2,2}}{z_3 - z_0} = \frac{4 - 2.87312}{0.5 - 0} = 2.25376.$$

z	$f(z)$	1st d.d	2nd d.d	3rd d.d
$z_0=0$	$1=Q_{0,0}$			
		$2=Q_{1,1}$		
$z_1=0$	$1=Q_{1,0}$		$2.87312=Q_{2,2}$	
		$3.43656=Q_{2,1}$		$2.25376=Q_{3,3}$.
$z_2=0.5$	$2.71828=Q_{2,0}$		$4=Q_{3,2}$	
		$5.43656=Q_{3,1}$		
$z_3=0.5$	$2.71828=Q_{3,0}$			

Hermite interpolating polynomial is

$$H(x) = Q_{0,0} + Q_{1,1}(x-x_0) + Q_{2,2}(x-x_0)^2 + Q_{3,3}(x-x_0)^2(x-x_1)$$

$$= 1 + 2(x-0) + 2.87312(x-0)^2 + 2.25376(x-0)^2(x-0.5)$$

$$= 1 + 2x + 2.87312x^2 + 2.25376x^2(x-0.5)$$

$$H(x) = 1 + 2x + 1.74624x^2 + 2.25376x^3.$$

(5) Use the following values and five digit rounding arithmetic to construct the Hermite interpolating polynomial to approximate $\sin(0.34)$.

x	$\sin x$	$D_x(\sin x)=\cos x$
0.30	0.29552	0.95534
0.32	0.31457	0.94924
0.35	0.34290	0.93937

Sol:- Given $x_0=0.30$, $x_1=0.32$, $x_2=0.35$

$f(x_0)=0.29552$, $f(x_1)=0.31457$, $f(x_2)=0.34290$

$f'(x_0)=0.95534$, $f'(x_1)=0.94924$, $f'(x_2)=0.93937$

- By Hermite interpolation $Q_{i,j} = \dfrac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$ $\rightarrow (1)$

where $i = 2, 3, \ldots 2n+1$
$j = 2, 3, \ldots i$

$z_0 = z_1 = x_0 = 0.3$

$z_2 = z_3 = x_1 = 0.32$ (Here $i = 0, 1, 2$)

$z_4 = z_5 = x_2 = 0.35$

- ~~$Q_{0,0} = f(x_0) = 0.295520$~~
~~$Q_{0,0}$~~

$Q_{0,0} = Q_{1,0} = f(x_0) = 0.29552$

$Q_{2,0} = Q_{3,0} = f(x_1) = 0.31457$

$Q_{4,0} = Q_{5,0} = f(x_2) = 0.34290$

$Q_{1,1} = f'(x_0) = 0.95534$

$Q_{3,1} = f'(x_1) = 0.94924$

$Q_{5,1} = f'(x_2) = 0.93937.$

from (1) $Q_{2,1} = \dfrac{Q_{2,0} - Q_{1,0}}{z_2 - z_1} = 0.9525$

$Q_{4,1} = \dfrac{Q_{4,0} - Q_{3,0}}{z_4 - z_3} = 0.9443333$

$Q_{2,2} = \dfrac{Q_{2,1} - Q_{1,1}}{z_2 - z_0} = -0.142$

$Q_{3,2} = \dfrac{Q_{3,1} - Q_{2,1}}{z_3 - z_1} = -0.163$

$Q_{4,2} = \dfrac{Q_{4,1} - Q_{3,1}}{z_4 - z_2} = -0.163556$

$Q_{5,2} = \dfrac{Q_{5,1} - Q_{4,1}}{z_5 - z_3} = -0.165443$

$Q_{3,3} = \dfrac{Q_{3,2} - Q_{2,2}}{z_3 - z_0} = -1.05$

$Q_{4,3} = \dfrac{Q_{4,2} - Q_{3,2}}{z_4 - z_1} = -0.01112$

$Q_{5,3} = \dfrac{Q_{5,2} - Q_{4,2}}{z_5 - z_2} = -0.0629$

$Q_{4,4} = \dfrac{Q_{4,3} - Q_{3,3}}{z_4 - z_0} = 20.776$

$Q_{5,4} = \dfrac{Q_{5,3} - Q_{4,3}}{z_5 - z_1} = -1.0356$

$Q_{5,5} = \dfrac{Q_{5,4} - Q_{4,4}}{z_5 - z_0} = -436.2$

z	$f(z)$	1st d.d	2nd d.d	3rd d.d	4th d.d	5th d.d
$x_0=z_0=0.3$	$Q_{0,0}=0.29552$					
		$Q_{1,1}=$				
$x_0=z_1=0.3$	$Q_{1,0}=$		$Q_{2,2}=$			
		$Q_{2,1}=$		$Q_{3,3}=$		
$x_1=z_2=0.32$	$Q_{2,0}=$		$Q_{3,2}=$		$Q_{4,4}$	
		$Q_{3,1}=$		$Q_{4,3}=$		$Q_{5,5}$
$x_1=z_3=0.32$	$Q_{3,0}=$		$Q_{4,2}=$		$Q_{5,4}$	
		$Q_{4,1}=$		$Q_{5,3}$		
$x_2=z_4=0.35$	$Q_{4,0}=$		$Q_{5,2}=$			
		$Q_{5,1}=$				
$x_2=z_5=0.35$	$Q_{5,0}=$					

By Hermite interpolation polynomial.

$$H_5(x)= Q_{0,0}+ Q_{1,1}(x-x_0)+Q_{2,2}(x-x_0)^2+Q_{3,3}(x-x_0)^2(x-x_1)+$$
$$Q_{4,4}(x-x_0)^2(x-x_1)^2+Q_{5,5}(x-x_0)^2(x-x_1)^2(x-x_2)$$

$$=0.29552+0.95534(x-0.3)+(-0.142)(x-0.3)^2+(-1.05)(x-0.3)^2(x-0.32)$$
$$+20.776(x-0.3)^2(x-0.32)^2+(-436.232)(x-0.3)^2(x-0.32)^2(x-0.35)$$

At $x=0.34$.

$$\boxed{H_5(0.34)=0.33348}$$

6. $f(x)=3xe^{x}-e^{2x}$, approximate $f(1.03)$ by the Hermite interpolating polynomial of degree atmost three using $x_0=1$ and $x_1=1.05$.

Sol:- Given $f(x) = 3xe^{x} - e^{2x}$

$f'(x) = 3xe^{x} + 3e^{x} - 2e^{2x}$

Given $x_0 = 1$, $x_1 = 1.05$

$f(x_0) = f(1) = 3(1)e^{1} - e^{2(1)} = 3e - e^{2} = 0.765789$

$f(x_1) = f(1.05) = 3(1.05)e^{1.05} - e^{2(1.05)} = 0.835431$

$f'(x_0) = f'(1) = 1.531579$

$f'(x_1) = f'(1.05) = 1.242214$

x	$f(x)$	$f'(x)$
$x_0 = 1$	$f(x_0) = 0.765789$	$f'(x_0) = 1.531579$
$x_1 = 1.05$	$f(x_1) = 0.835431$	$f'(x_1) = 1.242214$

By Hermite interpolating $Q_{i,j} = \dfrac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$ → (1)

$x_0 = z_0 = z_1 = 1$

$x_1 = z_2 = z_3 = 1.05$

$Q_{0,0} = Q_{1,0} = f(x_0) = 0.765789$

$Q_{2,0} = Q_{3,0} = f(x_1) = 0.835431$

$Q_{1,1} = f'(x_0) = 1.531579$

$Q_{3,1} = f'(x_1) = 1.242214$.

From (1) $Q_{2,1} = \dfrac{Q_{2,0} - Q_{1,0}}{z_2 - z_1} = 1.39284$

$Q_{2,2} = \dfrac{Q_{2,1} - Q_{1,1}}{z_2 - z_0} = -2.77478$

$Q_{3,2} = \dfrac{Q_{3,1} - Q_{2,1}}{z_3 - z_1} = -3.01252$

$Q_{3,3} = \dfrac{Q_{3,2} - Q_{2,2}}{z_3 - z_0} = -4.7548.$

By Hermite interpolation formula

$$H_3(x) = Q_{0,0} + Q_{1,1}(x-x_1) + Q_{2,2}(x-x_0)^2 + Q_{3,3}(x-x_0)^2(x-x_1)$$

$$H_3(x) = 0.765789 + 1.531579(x-1) + (-2.77478)(x-1)^2 +$$

$$(-4.7548)(x-1)^2(x-1.05)$$

substitute $x = 1.03$

$$H_3(1.03) = 0.8093246.$$